Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Mathematics and Mechanics 70 (2006) 516–526
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

# Resonant periodic motions of Hamiltonian systems with one degree of freedom when the Hamiltonian is degenerate☆

O.V. Kholostova

*Moscow, Russia*

Received 18 November 2004

**Abstract**

The motions of a non-autonomous Hamiltonian system with one degree of freedom which is periodic in time and where the Hamiltonian contains a small parameter is considered. The origin of coordinates of the phase space is the equilibrium position of the unperturbed or complete system, which is stable in the linear approximation. It is assumed that there is degeneracy in the unperturbed Hamiltonian when account is taken of terms no higher than the fourth degree (the frequency of the small linear oscillations depends on the amplitude) and, in this case, one of the resonances of up to the fourth order inclusive is realized in the system. Model Hamiltonians are constructed for each case of resonance and a qualitative investigation of the motions of the model system is carried out. Using Poincaré's theory of periodic motions and KAM-theory, a rigorous solution is given of the problem of the existence, bifurcations and stability of the periodic motions of the initial system, which are analytic with respect to fractional powers of the small parameter. The resonant periodic motions (in the case of the degeneracy being considered) of a spherical pendulum with an oscillating suspension point are investigated as an application.


## 1. Formulation of the problem

Consider the motions of a non-autonomous Hamiltonian system with one degree of freedom which is $2\pi$-periodic in time and where the Hamiltonian $H(x, p_x, t)$ contains a small parameter $\varepsilon$. The origin of coordinates $x = p_x = 0$ of the phase space is the equilibrium position of either the unperturbed system (when $\varepsilon = 0$) or the complete system (when $\varepsilon \neq 0$), which is stable in the linear approximation, and the function $H$ is analytic in its neighbourhood.

A theory of the non-linear oscillations of Hamiltonian systems with one degree of freedom when there are resonances of up to the fourth order inclusive has been constructed previously in a number of papers.[1–8] In particular, a rigorous solution of the problem of the existence, bifurcations and stability of the periodic motions of such systems (with a period which is a multiple of the period of the external perturbation), which are analytic in fractional and integral powers of the small parameter, has been obtained. The results are correct under the assumption that, in the unperturbed Hamiltonian, there is no degeneracy in terms of up to the fourth order inclusive in $x$ and $p_x$. This means that, in an unperturbed Hamiltonian having the form

$$\frac{1}{2}\omega(x^2 + p_x^2) + \frac{1}{4}c_2(x^2 + p_x^2)^2 + \dots$$

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 4, pp. 568–580, 2006.

*E-mail address:* markeev@ipmnet.ru.

0021-8928/$ – see front matter 
doi:10.1016/j.jappmathmech.2006.09.005

which has been normalized up to terms of the fourth order, the coefficient $c_2$ is non-zero. In this case, it is also sufficient to normalize the complete Hamiltonian up to terms of the fourth order, taking account of the available resonance and then, having studied the properties of the approximate (model) system, to draw conclusions regarding the motions of the complete system.

However, situations arise when solving specific problems when there is resonance in the system and, at the same time, for certain values of the parameters of the problem or the relations between them, the magnitude of $c_2$ is equal or close to zero. The frequency of small non-linear oscillations of the system is then independent of the amplitude when account is taken of terms of up to the fourth order inclusive in the Hamiltonian. Such a degenerate case leads to a qualitatively new situation and requires special treatment.

When investigating of a system where there is degeneracy in the Hamiltonian, it is necessary to take account of terms in $x$ and $p_x$ of up to the sixth order inclusive. The model Hamiltonians obtained as a result of normalization possess qualitatively new properties and do not reduce to known Hamiltonians.

The aim of this paper is to give a rigorous solution of the problem of the existence, bifurcation and stability of the periodic motions of a system for each case of resonance of up to the fourth order inclusive, when there is degeneracy of the unperturbed Hamiltonian. It is assumed that the Hamiltonian system being considered is close to an autonomous system. The origin of coordinates $x=p_x=0$ is the equilibrium position of the unperturbed system (in the case of resonance in forced oscillations) or the complete system (in the remaining cases of resonance) in the neighbourhood of which the Hamiltonian is analytic and is represented in the form

$$\begin{aligned} H(x, p_x, t; \varepsilon) &= H^{(0)}(x, p_x) + \varepsilon H^{(1)}(x, p_x, t) + \varepsilon^2 H^{(2)}(x, p_x, t) + \dots \\ H^{(0)}(x, p_x) &= H_2^{(0)} + H_3^{(0)} + H_4^{(0)} + \dots \\ H^{(k)}(x, p_x, t) &= H_1^{(k)} + H_2^{(k)} + H_3^{(k)} + H_4^{(k)} + \dots \end{aligned} \tag{1.1}$$

where $H_l^{(k)}$ are forms of the $i$-th degree in $x$ and $p_x$ with constant coefficients (when $k=0$) or coefficients which are $2\pi$-periodic in time (when $k\geq 1$). The forms $H_1^{(k)} (k \geq 1)$ of the first degree are only present in the last relation of (1.1) in the case of resonance in forced oscillations and, in the remaining resonance cases, $H_1^{(k)} \equiv 0$.

We shall assume that the unperturbed Hamiltonian $H^{(0)}$ has already been reduced to normal form up to terms in $x$ and $p_x$ of the sixth order inclusive and has the form

$$H^{(0)} = \frac{1}{2}\omega(x^2 + p_x^2) + \frac{1}{4}c_2(x^2 + p_x^2)^2 + \frac{1}{8}c_3(x^2 + p_x^2)^3 + \dots \tag{1.2}$$

where it is also assumed that $c_2 \simeq 0$ and $c_3 \neq 0$.

The following resonance cases are considered for a system with Hamiltonian (1.1), (1.2): the resonance in forced oscillations when the natural frequency $\omega$ of the small oscillations of the unperturbed system is close to an integer; the parametric resonance when the quantity $2\omega$ is close to an odd integer, and cases of resonance of the third and fourth orders when the system which has been linearized in the neighbourhood of the origin of coordinates is stable in the Lyapunov sense, its characteristic exponents $\pm i\lambda$ are pure imaginary and, at the same time, one of the quantities $3\lambda$ or $4\lambda$ is close to an integer (and there are no resonances of lower orders).

The non-linear oscillations of the system in a sufficiently small neighbourhood of the origin of coordinates (the magnitude of which depends on the order of the resonance) are investigated for each of the resonance cases which have been described. In Section 2, the Hamiltonian is transformed using a number of canonical replacements of the variables and it is reduced to a form which is characteristic of the resonance under consideration; the model part of the Hamiltonian is separated out. In Section 3, a qualitative investigation of the model systems is carried out, the equilibrium positions are described and the phase portraits are constructed. In Section 4, the problem of the existence, bifurcation and stability of the periodic motions of the complete system, which are analytic with respect to fractional powers of the small parameter, is investigated using Poincaré's theory of periodic motions and KAM-theory. The resonant periodic motions (when the degeneracy being considered is present) of a spherical pendulum, the suspension point of which executes vertical harmonic oscillations of small amplitude, are investigated as an application in Section 5.

## 2. Transformation of the Hamiltonian

Suppose there is an $n$-th order resonance ($n = 1, 2, 3$ or 4) in the system, where the cases when $n = 1$ and 2 correspond to a resonance in the forced oscillations and parametric resonance. We will assume that $n\omega \sim N(n = 1, 2)$ or $n\lambda \sim N(n = 3, 4)$, where $N$ is an integer.

Using a sequence of canonical replacements of the variables, we reduce the Hamiltonian (1.1), (1.2) to normal form. When $n = 2, 3, 4$, it is necessary to put $H_1^{(k)} \equiv 0$ in expressions (1.1).

We now make the change of variables

$$x = \varepsilon^{s_n}\xi, \quad p_x = \varepsilon^{s_n}\eta; \quad s_n = (6-n)^{-1}$$

in the Hamiltonian.

In the new variables, the Hamiltonian becomes

$$H = \frac{1}{2}\omega(\xi^2+\eta^2) + \frac{1}{4}\varepsilon^{2s_n}c_2(\xi^2+\eta^2)^2 + \frac{1}{8}\varepsilon^{4s_n}c_3(\xi^2+\eta^2)^3 + \varepsilon^{4s_n}h_n^{(1)}(\xi, \eta, t) + O(\varepsilon^{5s_n})$$

where $h_n^{(1)}(\xi, \eta, t)$, which is the form $H_n^{(1)}(x, p_x, t)$ from (1.1) in which the quantities $x$ and $p_x$ have been replaced by $\xi$ and $\eta$.

We now change to the polar coordinates $\varphi$ and $r$ using the formulae

$$\xi = \sqrt{2r}\sin\varphi, \quad \eta = \sqrt{2r}\cos\varphi$$

Then, using the replacement of variables

$$\varphi = \tilde{\varphi} + O(\varepsilon^{4s_n}), \quad r = \tilde{r} + O(\varepsilon^{4s_n})$$

which is $2\pi$-periodic in time and close to an identity replacement, we simplify the terms in $\xi$ and $\eta$ up to the sixth order inclusive, taking account of the resonance which is present. All terms with non-resonant harmonics are eliminated here, the coefficient $\omega$ in the quadratic part of the unperturbed Hamiltonian acquires a “correction” to the order of $\varepsilon$ when $n \geq 2$ and only terms with resonant harmonics remain in the terms of the $n$-th power in $\xi$ and $\eta$. The Hamiltonian of the system becomes

$$H = \hat{\lambda}r + \varepsilon^{2s_n}c_2r^2 + \varepsilon^{4s_n}c_3r^3 + \varepsilon^{4s_n}\kappa_n r^{n/2}\cos(n\varphi - Nt + n\varphi_0) + O(\varepsilon^{5s_n})$$

where $\hat{\lambda} = \omega$ when $n = 1$ and $\hat{\lambda} = \omega + O(\varepsilon) = \text{const}$ when $n \geq 2$. The resonance coefficient $\kappa_n$ is assumed to be positive, and this can always be achieved by means of a shift in $\tilde{\varphi}$.

We now put $c_2 = \varepsilon^{2s_n}a_2$, $\lambda = N/n - \varepsilon^{4s_n}\beta$ and make the change of variables

$$n\varphi - Nt + n\varphi_0 = n\tilde{\varphi}, \quad r = \tilde{r}$$

At the same time, the orders of the terms in the main part of the Hamiltonian are equalized:

$$\tilde{H} = \varepsilon^{4s_n}(-\beta\tilde{r} + a_2\tilde{r}^2 + c_3\tilde{r}^3 + \kappa_n\tilde{r}^{n/2}\cos n\tilde{\varphi}) + O(\varepsilon^{5s_n})$$

We now make the further replacement

$$\tilde{\varphi} = \sigma\theta + \frac{(1-\sigma)\pi}{2n}, \quad \tilde{r} = k_n\rho; \quad \sigma = \text{sign}\,c_3, \quad k_n = \left(\frac{\kappa_n}{|c_n|}\right)^{2s_n}$$

change to the new independent variable $\tau = (\varepsilon^4|c_3|^{2-n}\kappa_n^4)^{s_n}t$ and introduce the parameters

$$\mu = \sigma\beta(|c_3|^{n-2}\kappa_n^{-4})^{s_n}, \quad \nu = \sigma a_2(|c_3|^{n-4}\kappa_n^{-2})^{s_n}$$

The Hamiltonian becomes

$$\Gamma^{(n)} = \gamma_0^{(n)}(\theta, \rho) + \varepsilon^{s_n}\gamma_1^{(n)}(\theta, \rho^{1/2}, \tau; \varepsilon^{s_n}) \tag{2.1}$$

$$\gamma_0^{(n)} = -\mu\rho + \nu\rho^2 + \rho^3 + \rho^{n/2}\cos n\theta \qquad (2.2)$$

The function $\gamma_0^{(n)}(n = 1, \ldots, 4)$ in relation (2.2) is the model Hamiltonian for the case of $n$-th order resonance in the case of degeneracy of the unperturbed Hamiltonian, and the parameters $\mu$ and $\nu$ can take arbitrary values. The function $\gamma_1^{(n)}$ in relation (2.1) is $2\pi$-periodic in $\theta$, periodic in $\tau$ with a period $T \sim \varepsilon^{4s_n}$ and analytic with respect to all the variables in the domain $0 < \rho \ll 1$.

A Hamiltonian, similar to (2.2) when $n = 2$ and the larger part of the phase portraits corresponding to it, which are presented below in Section 3 were obtained earlier* in the treatment of a particular problem, the investigation of parametrically excited wave motions of a liquid in cases close to the critical case.

The differential equations of motion corresponding to Hamiltonian (2.1) have the form

$$\frac{d\theta}{d\tau} = -\mu + 2\nu\rho + 3\rho^2 + \frac{n}{2}\rho^{(n-2)/2}\cos n\theta + O(\varepsilon^{s_n}), \quad \frac{d\rho}{d\tau} = n\rho^{n/2}\sin n\theta + O(\varepsilon^{s_n}) \qquad (2.3)$$

## 3. Investigation of model systems

We will now carry out a qualitative investigation of the motions of systems which are described by the model Hamiltonians (2.2).

### 3.1. Equilibrium positions

*3.1.1.* In the case of resonance in forced oscillations, the equilibrium positions of a system with Hamiltonian (2.2) (when $n = 1$) are found from the relations

$$\sin\theta = 0, \quad f_1(r) = 6r^5 + 4\nu r^3 - 2\mu r + \cos\theta = 0 \ (r = \sqrt{\rho} \geq 0)$$

Simple analysis of the function $f_1(r)$ when $\theta = 0$ or $\theta = \pi$ shows that the number of zeroes of this function changes on passing through the values of the parameters $\nu$ and $\mu$ for which the following conditions are satisfied

$$f_1|_{\theta=0} = f'_{1r} = 0 \quad \text{or} \quad f_1|_{\theta=\pi} = f'_{1r} = 0$$

These conditions occur for values of the parameters $\nu$ and $\mu$ belonging to the curves represented by the solid lines in Fig. 1, *f*, which separate the plane of the parameters of the problem into four domains I–IV. The points $A_1$, $B_1$, $C_1$ and $D_1$ of intersection of these curves with the coordinate axes and with one another have the coordinates $(-5\cdot 2^{-9/5}, 0)$, $(0, 5\cdot 2^{-12/5}\cdot 3^{1/5})$, $(-5\cdot 2^{-8/5}, -5\cdot 2^{-16/5})$, $(-2.776\ldots, -2.055\ldots)$ respectively.

In domains I and IV in Fig. 1, *f*, the model system has three equilibrium positions, two of which are stable and one is unstable. In domain III, there are five equilibrium positions (three stable and two unstable) and, in domain II, there is a single stable equilibrium position.

*3.1.2.* In the case of parametric resonance of a system with Hamiltonian (2.2) (when $n = 2$) there is an equilibrium position $\rho = 0$ at the origin of coordinates, which is stable when $|\mu| > 1$ and unstable when $|\mu| < 1$. When $\mu = 1$, this equilibrium position is stable when $\nu < 0$ and unstable when $\nu \geq 0$, and, when $\mu = -1$, it is stable when $\nu \geq 0$ and unstable when $\nu < 0$.

The other equilibrium positions of the system are determined from the relations

$$\sin 2\theta = 0, \quad f_2(\rho) = 3\rho^2 + 2\nu\rho + (\cos 2\theta - \mu) = 0$$

In the plane of the parameters $\nu$ and $\mu$, we distinguish the domains I–IV (Fig. 2, *i*), for which the lines $\mu = 1$ and $\mu = -1$ serve as the boundaries and, also, the parts of the parabolae $\mu = 1 - \nu^2/3$ and $\mu = -1 - \nu^2/3$ when $\nu \leq 0$. In

* *Bordakov GA, Karpov II, Leonov VV, Sekerzh-Zen'kovich SYa, Shingareva IK.* Approximate solution of the problem of the parametric excitation of surface waves for a depth of liquid close to the critical depth. Preprint Inst Problem Mekh Ross Akad Nauk 1993; No. 526.

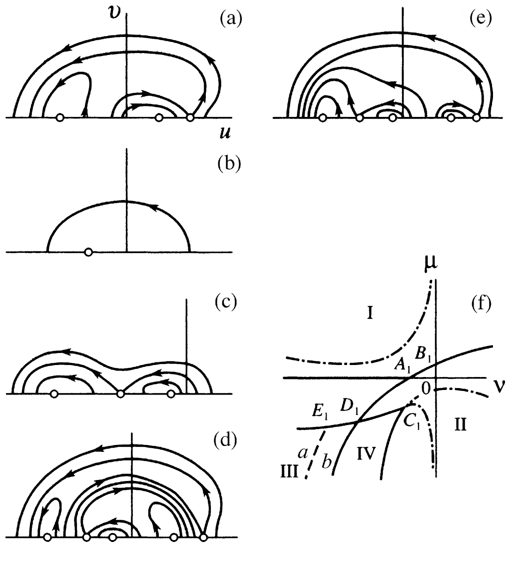


Fig. 1.

domain I, the equation $f_2(\rho) = 0$ has a single solution $\rho = \rho_4$ when $\cos 2\theta = -1$. In domain II, it has two solutions, $\rho = \rho_1$ and $\rho = \rho_2$, when $\cos 2\theta = 1$ and one solution $\rho = \rho_4$ when $\cos 2\theta = -1$. In domain III, it has two solutions, $\rho = \rho_1$ and $\rho = \rho_2$, when $\cos 2\theta = 1$ and two solutions, $\rho = \rho_3$ and $\rho = \rho_4$, when $\cos 2\theta = -1$. In domain IV, it has two solutions, $\rho = \rho_3$ and $\rho = \rho_4$, when $\cos 2\theta = -1$. In domain V, it has a single solution $\rho = \rho_4$ when $\cos 2\theta = -1$, and the equation

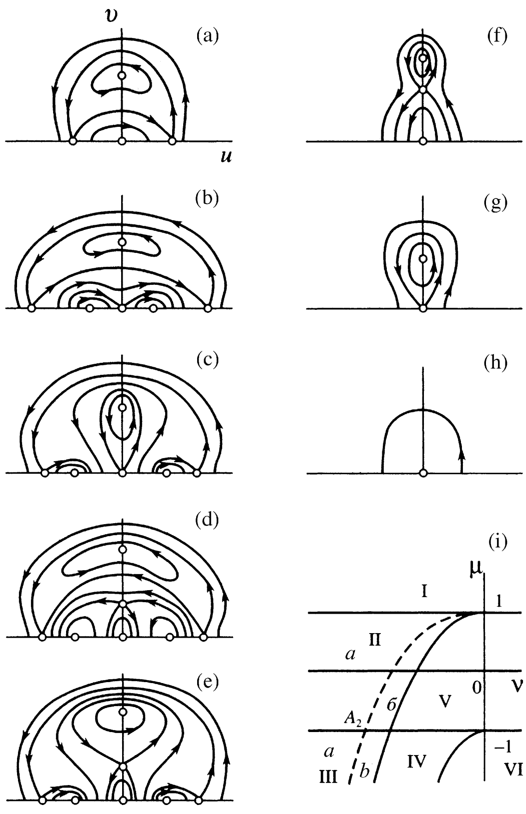


Fig. 2.

$f_2(\rho) = 0$ does not have positive solutions in domain VI. The following notation has been introduced here.

$$\rho_{1,2} = R^{\pm}(\nu, \mu - 1)(\rho_2 \geq \rho_1), \ \rho_{3,4} = R^{\pm}(\nu, \mu + 1)(\rho_4 \geq \rho_3); \ R^{\pm}(\nu, \mu) = \frac{-\nu \pm \sqrt{\nu^2 + 3\mu}}{3}$$

Stable equilibrium positions of the system correspond to the equilibrium values $\rho = \rho_1$ and $\rho = \rho_4$, and unstable equilibrium positions of the system to $\rho = \rho_2$ and $\rho = \rho_3$.

*3.1.3.* In the case of third-order resonance, a system with Hamiltonian (2.2) (when $n = 3$) has an equilibrium position $\rho = 0$ which is stable when $\mu \neq 0$ and unstable when $\mu = 0$. The other equilibrium positions are given by the equations

$$\sin 3\theta = 0, \quad f_3(r) = 3r^4 + 2\nu r^2 + \frac{3}{2} r \cos 3\theta - \mu = 0 \ (r = \rho^{1/2} \geq 0)$$

The number of equilibrium positions of the model system changes on passing through the values of the parameters $\nu$ and $\mu$ for which the following conditions are satisfied

$$f_3\big|_{\cos 3\theta = 1} = f'_{3r}\big|_{\cos 3\theta = 1} = 0 \quad \text{or} \quad f_3\big|_{\cos 3\theta = -1} = f'_{3r}\big|_{\cos 3\theta = -1} = 0$$

The curves corresponding to these conditions (shown by the solid lines in Fig. 3, *k*) and, also, the line $\mu = 0$ separate the plane of the parameters $\nu$ and $\mu$ into the domains I–V. The points $A_3$, $B_3$ and $C_3$ in Fig. 3, *k* have the coordinates $(-9 \cdot 2^{-7/3}, 0)$, $(0, -9/16)$ and $(-9 \cdot 2^{-8/3}, 9 \cdot 2^{-16/3})$ respectively.

In domain I, the equation $f_3(r) = 0$ has a single solution when $\cos 3\theta = 1$ and a single solution when $\cos 3\theta = -1$ to which the unstable and stable equilibrium positions of the model system correspond. In domain II, this equation has three solutions $r = r_1, r_2, r_3 (r_1 < r_2 < r_3)$ when $\cos 3\theta = 1$ and one solution when $\cos 3\theta = -1$: unstable equilibrium positions correspond to the solutions $r = r_1$ and $r_3$ and stable equilibrium positions to the solutions $r = r_2$ and $r_4$. In domain III, the equation $f_3(r) = 0$ has two solutions $r = r_1, r_2 (r_1 < r_2)$ when $\cos 3\theta = 1$ and two solutions $r = r_3$,

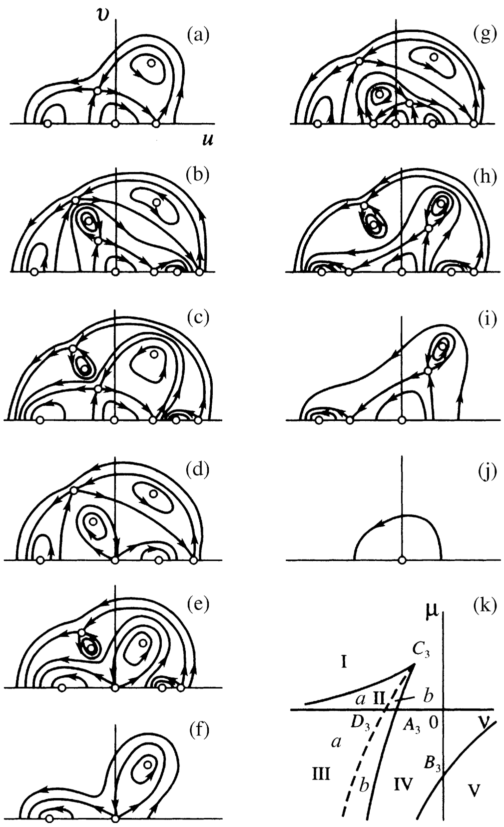


Fig. 3.

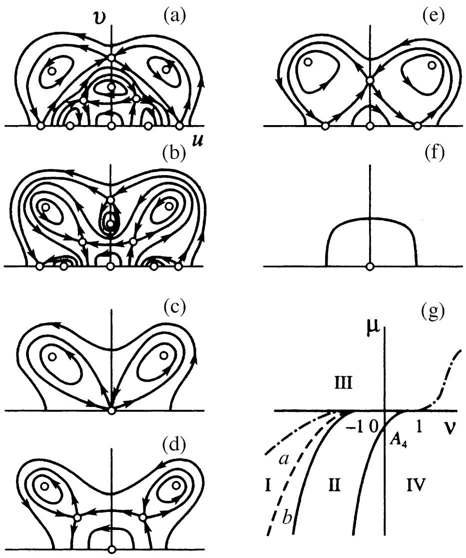


Fig. 4.

$r_4 (r_3 < r_4)$ when $\cos 3\theta = -1$. Stable equilibrium positions of the system correspond to the solutions $r = r_1$ and $r = r_4$, and unstable equilibrium positions of the system to $r = r_2$ and $r = r_3$. In domain IV, the equation $f_3(r) = 0$ has two solutions, $r = r_1$ and $r = r_2 (r_1 < r_2)$, when $\cos 3\theta = -1$ to which unstable and stable equilibrium positions of the system correspond. The equation does not have any solutions in domain V when $\cos 3\theta = \pm 1$.

*3.1.4* In the case of fourth-order resonance, a system with Hamiltonian (2.2) (when $n = 4$) has an equilibrium position $\rho = 0$ which is unstable when $\mu = 0$ and $-1 \leq \nu < 1$ and stable in the remaining cases.

In the plane of the parameters $\nu$ and $\mu$, we distinguish the domains I–IV (Fig. 4, *g*) for which the line $\mu = 0$ and parts of the parabolae $\mu = -(\nu + 1)^2/3$ (when $\nu \leq -1$) and $\mu = -(\nu - 1)^2/3$ (when $\nu \leq 1$) serve as the boundaries. Apart from the equilibrium position $\rho = 0$, in domain I the system also has equilibrium positions when $\rho = \rho_1$, $\rho_2$ and $\cos 4\theta = 1$ and equilibrium positions $\rho = \rho_3$, $\rho_4$ and $\cos 4\theta = -1$, where

$$\rho_{1,2} = R^{\pm}(\nu + 1, \mu), \quad \rho_{3,4} = R^{\pm}(\nu - 1, \mu), \quad \rho_3 < \rho_1 \leq \rho_2 < \rho_4$$

In domain II, the system has equilibrium positions when $\rho = \rho_3$, $\rho_4$ and $\cos 4\theta = -1$. In domain III, the system has equilibrium positions when $\rho = \rho_2$ and $\cos 4\theta = 1$ and equilibrium positions when $\rho = \rho_4$ and $\cos 4\theta = -1$. In domain IV, the system does not have equilibrium positions which differ from $\rho = 0$. Stable equilibrium positions of the model system correspond to the equilibrium values $\rho = \rho_1$ and $\rho = \rho_4$, and unstable equilibrium positions of the model system to $\rho = \rho_2$ and $\rho = \rho_3$.

### *3.2. Phase portraits*

The phase portraits of model systems with Hamiltonian (2.2) when $n = 1, \ldots, 4$ are shown in Figs. 1–4 respectively in the plane of the variables $u = \sqrt{2\rho}\cos\theta$, $v = \sqrt{2\rho}\sin\theta$ when $v \geq 0$. In the case of *n*-the order resonance, the phase portraits are converted into one another on rotating the plane about the origin of coordinates by an angle which is a multiple of $\pi/n$.

In Figs. 1–4, the singular points of a "centre" type correspond to stable equilibrium positions of the model systems, and saddle singular points correspond to unstable equilibrium positions. Merging of the two equilibrium points of the system, the stable and unstable points, into a single unstable complex singular point, which disappears on passing through the curve, occurs on the curvilinear boundaries of the domains. The corresponding phase portraits are not shown. The separatrices, separating the domains with different forms of motion of the system, pass through the unstable singular points. Each of the domains in which there are two types of unstable singular points (and this means two separatrices in the phase portraits) are subdivided into two subdomains, labelled with the letters *a* and *b*, with a different form of separatrices. An adjustment of the phase portraits, when two separatrices merge into one, occurs on the curves shown by the dashed lines in Fig. 1, *f*, Fig. 2, *i*, Fig. 3, *k* and Fig. 4, *g* (the corresponding phase portraits are not shown). The points $E_1$, $A_2$ and $D_3$ of intersection of these curves with the boundaries of the domain have the coordinates $(-3.485\ldots, -2.237\ldots)$, $(-2\sqrt{2}, -1)$ and $(-1.89\ldots, 0)$ respectively. Either oscillations in the neighbourhood of the stable equilibrium positions or rotations that are the closed trajectories in Figs. 1–4 encompassing the separatrices, correspond to the other trajectories of the model system. The phase portraits in the rectilinear boundaries of the domains are only shown in those cases when they are qualitatively different from the phase portraits in the domains adjacent to them.

Figure 1, *a–e* corresponds to the case of resonance in forced oscillations and to the domains I, II, IV, III*a* and III*b* in Fig. 1, *f*.

Figure 2, *a–h* corresponds to the case of parametric resonance and the domains I, II*a*, II*b*, III*b*, IV, V and VI in Fig. 2, *i*.

The phase portraits of the model system for third-order resonance are shown in Fig. 3. Figure 3, *a–c* corresponds to the domains I, II *a* and II *b*, Fig. 3, *d–f* corresponds to segments of the line $\mu = 0$ when $\nu < -1.89\ldots$, $-1.89\ldots < \nu < -9/8\cdot 2^{2/3}$, $\nu > -9/8\cdot 2^{2/3}$, and Fig. 3, *g–j* corresponds to the domains III *a*, III *b*, IV and V (see Fig. 3, *k*).

The phase portraits of the model system in the case of fourth-order resonance, shown in Fig. 4, *a–f*, correspond to the domains I *a* and I *b*, to the segment $-1 \le \nu < 1$ of the line $\mu = 0$ and to the domains II, III and IV in Fig. 4, *g*.

## 4. Periodic motions

We will now consider the complete system (2.3) when $n = 1, \ldots, 4$. In the neighbourhood of each equilibrium position $\theta = \theta_*$, $\rho = \rho_*$ of the model system, which does not coincide with the origin of the coordinates (we exclude the unstable complex singular points on the boundaries of the domains from the treatment), the complete system in the case of an $n$-th order resonance can be treated as a quasilinear system with perturbations of the order of $\varepsilon^{s_n}$ with period $T \sim \varepsilon^{4s_n}$ with respect to $\tau$ (see Section 2). According to Poincaré's theory of periodic motions,[9] a unique solution of the complete system which is $T$-periodic in $\tau$, analytic in $\varepsilon^{s_n}$ and has the form

$$\theta = \tilde{\theta}(\tau) = \theta_* + O(\varepsilon^{s_n}), \quad \rho = \tilde{\rho}(\tau) = \rho_* + O(\varepsilon^{s_n})$$

is generated from each equilibrium position $\theta = \theta_*$, $\rho = \rho_*$ of the model system.

In the initial system, the solution, which is analytic in $\varepsilon^{s_n}$, $2\pi n$-periodic in time and has the form

$$\begin{aligned} x &= \varepsilon^{s_n}\sqrt{2k_n\rho_*}\sin\psi_n + O(\varepsilon^{s_n}), \quad p_x = \varepsilon^{s_n}\sqrt{2k_n\rho_*}\cos\psi_n + O(\varepsilon^{2s_n}) \\ \psi_n &= Nt/n + \sigma\theta_* + (1-\sigma)\pi/(2n) - \varphi_0 \end{aligned} \tag{4.1}$$

corresponds to it.

Note that solutions of the form (4.1) are generated from the equilibrium positions, corresponding to the same values of $\rho_*$ and different values of $\theta_*$ (differing by $2\pi/n$, $n \ge 2$), which transfer into one another for a shift in $t$ by $2\pi, 4\pi, \ldots, 2(n-1)\pi$ and correspond to the same motion of the initial system.

We will now consider the problem of the stability of these solutions. The periodic solutions, which are generated from unstable equilibrium positions of the model system, are unstable, which follows from the continuity with respect to $\varepsilon$ of the characteristic exponents of the corresponding linear equations of the perturbed motion.

The problem of the stability of the periodic solutions generated from stable equilibrium positions of the model system can be solved by normalization of the complete Hamiltonians in the neighbourhood of these solutions. In the complete Hamiltonian, we put $\theta = \tilde{\theta}(\tau) + q$, $\rho = \tilde{\rho}(\tau) + p$ and, then using the canonical transformation $q, p \to \tilde{q}, \tilde{p}$, we reduce the Hamiltonian to normal form up to terms of the fourth order inclusive

$$\tilde{\Gamma} = \frac{1}{2}\left(\omega_* + O(\varepsilon^{s_n})\right)(\tilde{q}^2 + \tilde{p}^2) + \frac{1}{4}\left(c_* + O(\varepsilon^{s_n})\right)(\tilde{q}^2 + \tilde{p}^2)^2 + O_5 \tag{4.2}$$

The coefficients $\omega_* + O(\varepsilon^{s_n})$ and $c_* + O(\varepsilon^{s_n})$ in the terms of the second and fourth powers in Hamiltonian (4.2) are constant quantities. If the coefficient $c_*$ is non-zero, then, for sufficiently small values of $\varepsilon$, the periodic motion being considered is stable, by the Arnold–Moser theorem.[10]

Calculations show that the condition of this theorem is violated (the coefficient $c_*$ vanishes) only in the cases of resonance in forced oscillations and fourth-order resonance for values of the parameters $\nu$ and $\mu$ belonging to the curves represented by the dash-dot lines in Fig. 1, *f* and Fig. 4, *g*. In Fig. 1, *f*, the curve from domain I corresponds to the periodic solution generated from the stable equilibrium position $\theta_* = 0$, $\rho = \rho_*$ (see Fig. 1, *a*), and the two curves from domain II correspond to the periodic motion generated from the stable equilibrium position $\theta_* = \pi$, $\rho = \rho_*$, which is unique in this domain (see Fig. 1, *b*).In Fig. 4, *g*, the curve from the domain I *a* corresponds to the periodic motion generated from the stable equilibrium position when $\rho = \rho_1$ and $\cos 4\theta = 1$ (see Fig. 4, *a*) and the curve from domain III corresponds to the periodic motion generated from the stable equilibrium position when $\rho = \rho_4$ and $\cos 4\theta = -1$ (see Fig. 4, *e*).

Outside the above-mentioned curves for these periodic solutions and, also, for all possible values of the parameters of the remaining periodic solutions generated from the equilibrium positions of the model system of the two resonance cases mentioned above and all remaining resonance cases, the corresponding coefficient $c_*$ of the normal form of the perturbed Hamiltonian keeps a constant sign and, consequently, the periodic solutions are stable.

## 5. Application: a spherical pendulum with a vibrating suspension point

We will now consider the motion of a spherical pendulum of mass $m$ and length $l$, the suspension point of which executes vertical harmonic oscillations of small amplitude of the form $\varepsilon l \cos \Omega t (0 < \varepsilon \ll 1)$ relative to a certain fixed point in space. We will describe the position of the pendulum using the spherical coordinates $\theta$ and $\varphi$ and we will denote the momenta corresponding to them by $p_\theta$ and $p_\varphi$.

The motions of the pendulum are described by canonical equations with Hamiltonian

$$H = \frac{1}{2ml^2}\left[(p_\theta - \varepsilon m l^2 \Omega \sin\Omega t \sin\theta)^2 + \frac{p_\varphi^2}{\sin^2\theta}\right] - mgl\cos\theta$$

The coordinate $\varphi$ is cyclic, and therefore $p_\varphi = \text{const}$. We put

$$p_\varphi = ml^2\sin^2\theta_0\omega_0, \quad \theta_0 = \arccos(g/(\omega_0^2 l)), \quad \omega_0 = \text{const} > 0$$

We make the canonical replacement of variables

$$\theta = q, \quad p_\theta = ml^2\omega_0(p + \varepsilon\hat{\Omega}\sin\Omega t\sin q), \quad \hat{\Omega} = \Omega/\omega_0$$

and change to dimensionless time $\tau = \omega_0 t$. The Hamiltonian of the reduced system with one degree of freedom takes the form

$$H = \frac{1}{2}\left(p^2 + \frac{\sin^4\theta_0}{\sin^2 q}\right) - (\cos\theta_0 + \varepsilon\hat{\Omega}^2\cos\hat{\Omega}\tau)\cos q$$

We now make a further replacement of variables $q, p \to u, v$ using the formulae $u = \cos q$, $p = -v\sin q$ and write the Hamiltonian in the algebraic form

$$H = H_0 + \varepsilon H_1$$
$$H_0 = \frac{1}{2}(1-u^2)v^2 + \frac{1}{2}\frac{(1-u_0^2)^3}{1-u^2} - u_0 u, \quad H_1 = -\hat{\Omega}^2\cos\hat{\Omega}\tau u, \quad u_0 = \cos\theta_0 \tag{5.1}$$

The system with the unperturbed Hamiltonian $H_0$ corresponds to a pendulum with a fixed suspension point. It has a single equilibrium position $u = u_0$, $v = 0$ to which conical motion of the pendulum corresponds, when it is deflected from the vertical by an angle $\theta_0$ and rotates about it at a constant angular velocity $\omega_0$.

The high-frequency oscillations of a pendulum close to its conical motion were investigated earlier in Ref. 11 in the case of small amplitude vertical high-frequency harmonic vibrations of the suspension point.

In the Hamiltonian $H_0$, we put

$$u = u_0 + \sqrt{\frac{1-u_0^2}{\omega}}\xi, \quad v = \sqrt{\frac{\omega}{1-u_0^2}}\eta, \quad \omega = \sqrt{1+3u_0^2}$$

Then, using a replacement of the variable $\xi = x + \ldots$, $\eta = p_x + \ldots$, which is close to an identity replacement, we transform the function $H_0$ to a normal form of the type (1.2), where

$$c_2 = -\frac{3u_0^4 + 15u_0^2 - 2}{4(1+3u_0^2)^2}$$

$$c_3 = \frac{40 + 1648u_0^2 + 5573u_0^4 + 37100u_0^6 + 52966u_0^8 + 34620u_0^{10} + 7317u_0^{12}}{48\omega^9(1-u_0^2)^2}$$

The coefficient $c_2$ vanishes at the point $u_0 = u_0^* = \sqrt{6\sqrt{249} - 90}/6 = 0.3605$, which corresponds to a deflection of the pendulum from the vertical by an angle $\theta_0 = \theta_0^* = 68.895\ldots^\circ$. For this value of $u_0$, we have $c_3 = c_3^* = 2.791\ldots$.

We will now construct the periodic motions of a pendulum with a vibrating suspension point in the neighbourhood of its conical motion in the unperturbed problem, assuming that $c_2 \sim 0$ and $\hat{\Omega} \sim n\omega (n = 1, \ldots, 4)$. In the case of an $n$-th order resonance ($n = 1, \ldots, 4$), it is necessary to assume (see Section 2) that $c_2 \sim \varepsilon^{2s_n}$ and that the resonance frequency difference is of the order of $\varepsilon^{4s_n}$. The straight line $u_0 = u_0^*$ and the part of the curve $\hat{\Omega} = \hat{\Omega}(u_0)$ when $u_0 \in (-1; 1)$, where $\hat{\Omega}(u_0) = \omega(u_0) = \sqrt{1+3u_0^2}$ when $n = 1$ or $\hat{\Omega}(u_0) = n\lambda(u_0)(\lambda(u_0) = \omega(u_0) + O(\varepsilon^2))$ when $n = 2$, 3, 4, in the plane of the parameters $u_0$, $\hat{\Omega}$ of the problem corresponds to the condition $c_2 = 0$ and the exact resonance $\hat{\Omega} = \omega$ (when $n = 1$) or $\hat{\Omega} = n\lambda$ (when $n = 2, 3, 4$).

We now consider the neighbourhood of the point $(u_0^*, \hat{\Omega}(u_0^*))$ of their intersection (Fig. 5), which consists of points which are a distance $\sim \varepsilon^{2s_n}$ from the line $u_0 = u_0^*$ and a distance $\sim \varepsilon^{4s_n}$ from the curve $\hat{\Omega} = \hat{\Omega}(u_0)$, and is a curvilinear quadrilateral. In the case of resonance in forced oscillations ($n = 1$), for an arbitrary point $(u_0, \hat{\Omega})$ of this neighbourhood we put $u = u_0 + \tilde{u}$, $v = \tilde{v}$ in expression (5.1) and $u = \hat{u}(\tau) + \tilde{u}$, $v = \tilde{v}(\tau) + \tilde{v}$ in the remaining resonance cases, where

$$\hat{u}(\tau) = u_0 + \varepsilon\frac{(1-u_0^2)\hat{\Omega}^2}{\omega^2 - \hat{\Omega}^2}\cos\hat{\Omega}\tau + O(\varepsilon^2), \quad \hat{v}(\tau) = -\varepsilon\frac{\hat{\Omega}^3}{\omega^2 - \hat{\Omega}^2}\sin\hat{\Omega}\tau + O(\varepsilon^2) \tag{5.2}$$

Relations (5.2) describe the non-resonance forced oscillations of the reduced system with Hamiltonian (5.1), which are $2\pi$-periodic in $\tau$ and are generated, when $\varepsilon \neq 0$, from the equilibrium position $u = u_0$, $v = 0$ of the unperturbed system.

On then carrying out transformations of the Hamiltonian, similar to those described in Section 2, for each resonance case, we obtain a Hamiltonian in the form of (2.1), where it is necessary to put $N = 1$, $t = \hat{\Omega}\tau$, $\lambda = \omega + O(\varepsilon^2)$ and, also, as calculations show, $\varphi_0 = \pi/2$, $\kappa_1 = 0.844\ldots$ (for the resonance in forced oscillations), $\varphi_0 = 0$, $\kappa_2 = 0.532\ldots$ (for the parametric resonance), $\varphi_0 = -\pi/6$, $\kappa_3 = 0.141\ldots$ (for the third-order resonance) and $\varphi_0 = 0$, $\kappa_4 = 0.951\ldots$ (for the fourth-order resonance).

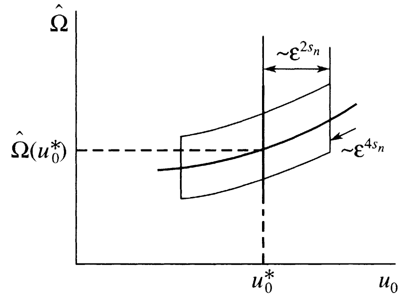


Fig. 5.

We now introduce the parameters $\nu$ and $\mu$ and consider a system with Hamiltonian (2.2) when $n = 1, 2, 3$, or 4. It follows from the results in Section 2 that the coordinates $(u_0, \hat{\Omega})$ of the point being considered are connected with the parameters $\nu$ and $\mu$ of the corresponding model system by relations of the form ($\alpha_n$, $\beta_n$, $\gamma_n$ are constants)

$$u_0 = u_0^* + \varepsilon^{2s_n}\alpha_n\nu + \dots, \quad \hat{\Omega} = \hat{\Omega}(u_0^*) + \varepsilon^{2s_n}\beta_n\nu + \varepsilon^{4s_n}\gamma_n\mu + \dots \tag{5.3}$$

Formulae (5.3) define the transformation of the plane of the parameters $\nu$ and $\mu$ of the model system into the neighbourhood of the point $(u_0^*, \hat{\Omega}(u_0^*))$ in the plane of the parameters $u_0$, $\hat{\Omega}$. As a result of this transformation, the domains in Fig. 1, *f*, Fig. 2, *i*, Fig. 3, *k* and Fig. 4, *g* become the corresponding domains of the neighbourhood being studied and, with an appropriate choice of the point $(u_0, \hat{\Omega})$, any of the cases described in Sections 3 and 4 can be realized.

For a specified point $(u_0, \hat{\Omega})$, the values of the parameters $\nu$ and $\mu$ are calculated and the equilibrium positions $\rho = \rho_*$, $\theta = \theta_*$ of the model system are determined. The periodic motions of the system with Hamiltonian (5.1) are then constructed and conclusions can be drawn regarding their stability. These periodic motions have the form (see Section 4)

$$u(\tau) = \hat{u}_0 + \varepsilon^{s_n}\sqrt{2n(1-u_0^{*2})k_n\rho_*}\sin\left(\frac{\hat{\Omega}\tau}{n} + \theta_* - \varphi_0\right) + O(\varepsilon^{2s_n}) \tag{5.4}$$

where $\hat{u}_0 = u_0$ when $n = 1$, $\hat{u}_0 = \hat{u}(\tau)$ when $n = 2, 3, 4$, and the quantities $k_n$ are defined in Section 2.

Relation (5.4) describes the change in the cosine of the angle of deflection of the pendulum from the vertical, and, at the same time, the angular velocity of rotation of the pendulum about the vertical is given by the relation

$$\frac{d\varphi}{d\tau} = \frac{1-u_0^2}{1-u^2(\tau)} \tag{5.5}$$

Formulae (5.4) and (5.5) give the resonant periodic motions of the pendulum in the neighbourhood of its conical motion when the unperturbed Hamiltonian is degenerate.

## Acknowledgement

This research was supported financially by the Russian Foundation for Basic Research (05-01-00386) and the Program for the Support of Leading Scientific Schools (NSh -1477.2003.1).

## References

1. Markeyev AP. Third-order resonance in a Hamiltonian system with one degree of freedom. *Prikl Mat Mekh* 1994;**58**(5):37–48.
2. Markeyev AP. The behaviour of a non-linear Hamiltonian system with one degree of freedom on the boundary of a domain of parametric resonance. *Prikl Mat Mekh* 1995;**59**(4):569–80.
3. Markeyev AP. Parametric resonance and non-linear oscillations of a heavy rigid body in the neighbourhood of its planar rotations. *Izv Ross Akad Nauk MTT* 1995;**5**:34–44.
4. Kholostova OV. The motion of a Hamiltonian system with one degree of freedom when there is resonance in forced oscillations. *Izv Ross Akad Nauk MTT* 1996;**3**:167–75.
5. Kholostova OV. Parametric resonance in the problem of the non-linear oscillations of a satellite in an elliptic orbit. *Kosmich Issled* 1996;**34**(3):312–6.
6. Markeyev AP. The critical case of fourth-order resonance in a Hamiltonian system with one degree of freedom. *Prikl Mat Mekh* 1997;**61**(3):369–76.
7. Kholostova OV. The non-linear oscillations of a satellite in the case of third-order resonance. *Prikl Mat Mekh* 1997;**61**(4):556–65.
8. Kholostova OV. The non-linear oscillations of a Hamiltonian system with one degree of freedom accompanying fourth-order resonance. *Prikl Mat Mekh* 1998;**62**(6):957–67.
9. Malkin IG. *Some Problems in the Theory of Non-linear Oscillations*. Moscow: Gostekhizdat; 1956.
10. Moser JK. *Lectures on Hamiltonian Systems*. Providence, R.I.: Amer. Math. Soc.; 1968.
11. Markeyev AP. The dynamics of a spherical pendulum with a vibrating support. *Prikl Mat Mekh* 1999;**63**(2):213–9.

*Translated by* E.L.S.